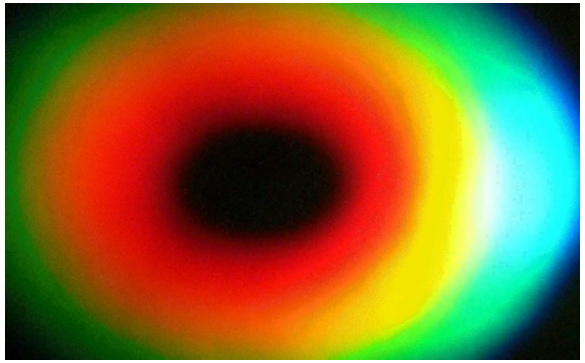
Lecture 12

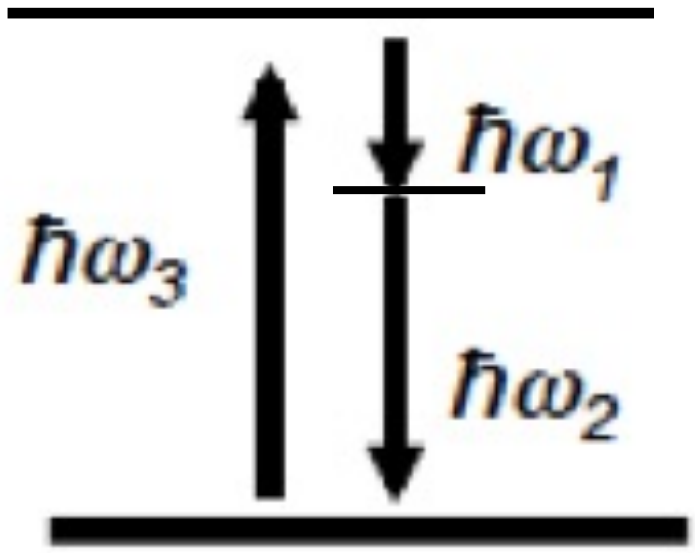


Optical parametric oscillators and amplifiers.

Singly- and doubly- resonant oscillators.

Optical parametric process

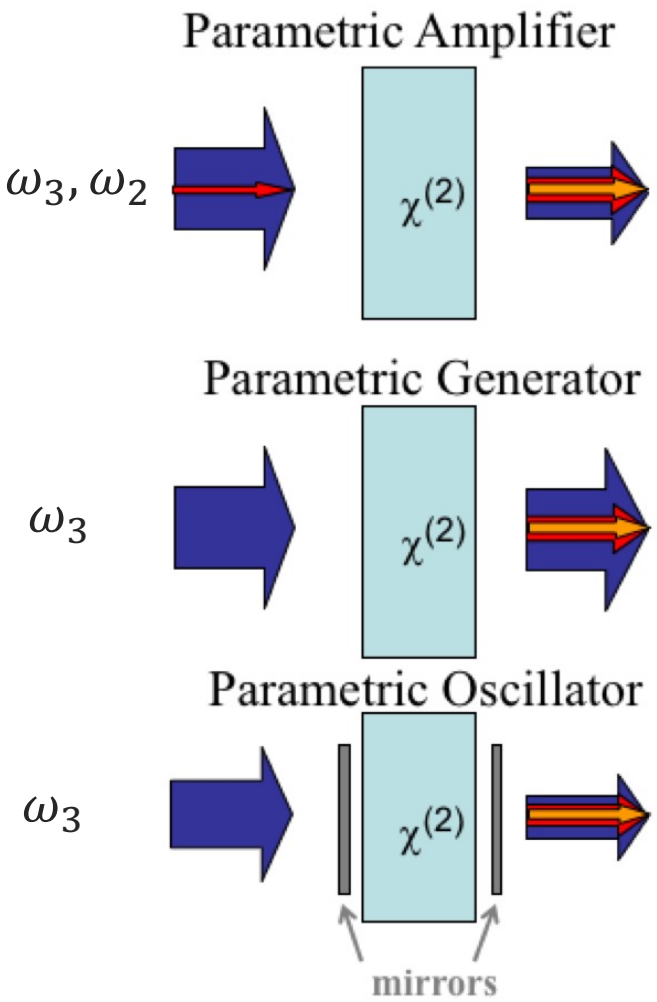
Only one strong input is present



$$\omega_3 = \omega_1 + \omega_2$$

Optical parametric processes

$$\omega_3 \rightarrow \omega_1 + \omega_2$$



Amplifier (OPA):
Weak signal at ω_1 is amplified, and ω_2 is generated

Generator (OPG):
No seed. Both ω_1 and ω_2 (signal and idler) are generated

Oscilaltor (OPO):
Both ω_1 and ω_2 (signal and idler) are generated through a resonant enhancement

Optical parametric amplification

Again, recall coupled-wave equations for 3 waves – Lecture 5 (5.7)

$$\boxed{\omega_3 = \omega_1 + \omega_2}$$

$$\frac{dA_1}{dz} = -ig\, A_3 A_2^* e^{-i\Delta kz}$$

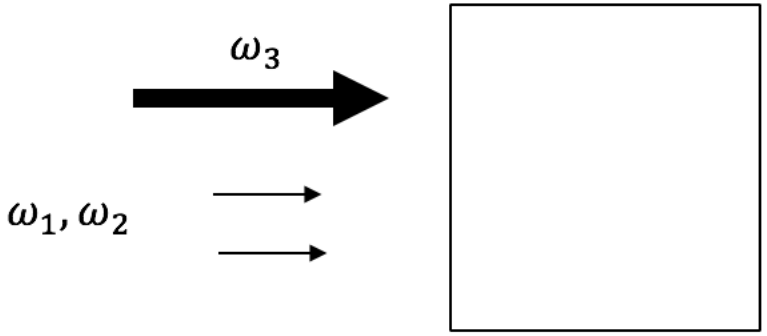
$$\frac{dA_2}{dz} = -ig\, A_3 A_1^* e^{-i\Delta kz} \qquad (5.7)$$

$$\frac{dA_3}{dz} = -igA_1 A_2 e^{i\Delta kz}$$

NL coupling coefficient

$$g = \frac{d}{c}\sqrt{\frac{\omega_1\omega_2\omega_3}{n_1 n_2 n_3}}$$

Now, assume that A_3 field ('pump') is much stronger than A_1 and A_2 fields ('signal' and 'idler'). We can also assume (by the proper choice of time origin) A_3 to be real: $A_3 = A_3^*$; we also assume A_3 is constant – undepleted pump approximation, Δk=0 (no phase mismatch), and no absorption.



define $\gamma = gA_3$ and get:

$$\frac{d}{dz}\Big|_{*} \quad \frac{dA_1}{dz} = -i\gamma A_2^* \qquad \frac{dA_2}{dz} = -i\gamma A_1^* \qquad (5.7a)$$

Optical parametric amplification

$$\rightarrow \quad \frac{d^2A_1}{dz^2} - \gamma^2 A_1 = 0 \quad \text{and} \quad \frac{d^2A_2}{dz^2} - \gamma^2 A_2 = 0$$

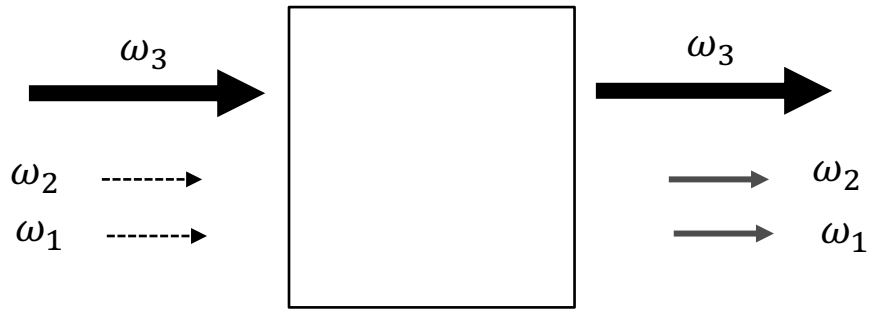
The general solutions to this equation are: $e^{\pm\gamma z}$ or $\sinh(\gamma z)$ & $cosh(\gamma z)$

$$\cosh x = \tfrac{1}{2}(e^{x} + e^{-x})$$
$$\sinh x = \tfrac{1}{2}(e^{x} - e^{-x})$$

Look for solutions in the form:

$$A_1 = a_1 cosh(\gamma z) + b_1 \sinh(\gamma z)$$
$$A_2 = a_2 cosh(\gamma z) + b_2 \sinh(\gamma z)$$

For initial conditions: $A_1 = A_{10}$
$A_2 = A_{20}$



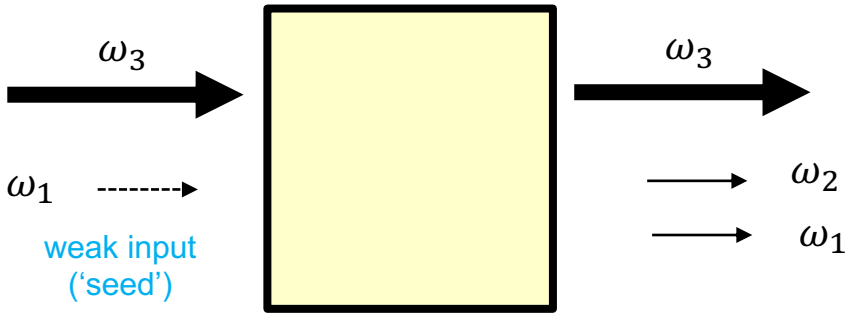
Solution:

$$A_1 = A_{10} cosh(\gamma z) - iA_{20}^* \sinh(\gamma z)$$
$$A_2 = A_{20} cosh(\gamma z) - iA_{10}^* \sinh(\gamma z) \qquad (12.1)$$

Optical parametric amplifier (OPA)

Now if we have input at only one wave:

Initial conditions: $A_1 = A_{10}$
$A_2 = 0$



Solution:

$$A_1 = A_{10}\, cosh(\gamma z)$$
$$A_2 = -iA_{10}^{*} \sinh(\gamma z) \qquad (12.2)$$

The **OPA** uses three-wave mixing in a nonlinear crystal to provide an optical gain of the 'seed' at ω_1. Simultneously, the wave at ω_2 grows from zero and is amplified.

The corresponding photon-flux densities are:

$$\Phi_1 = \Phi_{10} cosh^2(\gamma z)$$
$$\Phi_2 = \Phi_{10} sinh^2(\gamma z) \qquad (12.3)$$

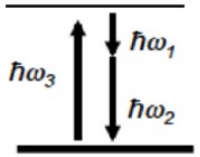
$$\Phi_1 - \Phi_{10} = \Phi_{10}(cosh^2(\gamma z) - 1) = \Phi_{10} sinh^2(\gamma z) = \Phi_2 \qquad (12.3a)$$

In terms of intensities:

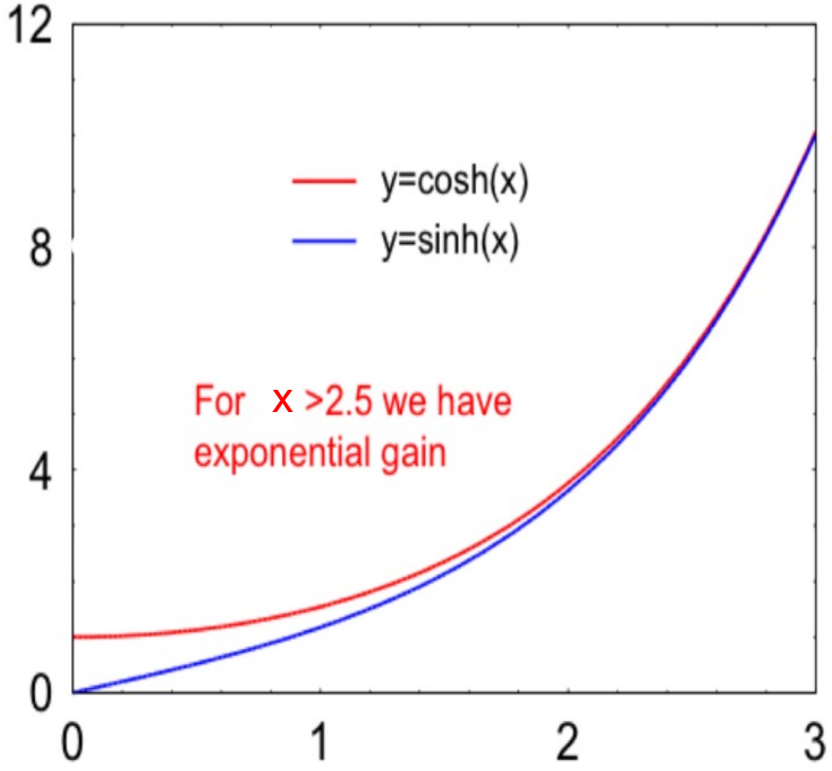
$$I_1 = I_{10}\, cosh^2(\gamma z)$$
$$I_2 = \frac{\omega_2}{\omega_1} I_{10}\, sinh^2(\gamma z) \qquad (12.4)$$

These expressions are symmetric under the interchange of ω_1 and ω_2

Optical parametric amplification



$\cosh x = \frac{1}{2}(e^{x} + e^{-x})$
$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$



For $\gamma z \gg 1$

$$\cosh^2(\gamma z) = \frac{1}{4}(e^{\gamma z} + e^{-\gamma z})^2 \approx e^{2\gamma z}/4 \qquad (12.5)$$

– the gain increases **exponenially** with γz. This behavior of monotonic growth of both waves is different from that of sum-frequency generation, where oscillatory behavior occurs.

Recalling that $I_\omega = \frac{c\varepsilon_0}{2}\omega|A_\omega|^2$ → $|A_\omega|^2 = \frac{2I_\omega}{c\varepsilon_0\omega}$

The gain coefficient

$$\boldsymbol{\gamma} = gA_3 = \frac{d}{c}\sqrt{\frac{\omega_1\omega_2\omega_3}{n_1n_2n_3}}A_3 = \sqrt{\frac{2\omega_1\omega_2}{\varepsilon_0c^3}\left(\frac{d^2}{n^3}\right)I_{\omega_3}}$$

$$= \sqrt{\frac{8\pi^2}{\lambda_1\lambda_2\varepsilon_0c}\left(\frac{d^2}{n^3}\right)I_{\omega_3}} \qquad (12.6)$$

Optical parametric process with some phase mismatch

Now, allow for the phase mismatch: $\Delta k \neq 0$ →

$$\frac{dA_1}{dz} = -i\gamma A_2^* e^{-i\Delta kz}$$
$$\frac{dA_2}{dz} = -i\gamma A_1^* e^{-i\Delta kz} \qquad (12.7)$$

Make a substitution: $A_1 = a_1 e^{-i\Delta kz/2}, A_2 = a_2 e^{-i\Delta kz/2}$

Then (12.7) is reduced (see Stegeman book p.114) to the equations

$$\frac{d^2a_1}{dz^2} - \gamma'^2 a_1 = 0 \qquad \frac{d^2a_2}{dz^2} - \gamma'^2 a_2 = 0 \qquad (12.8)$$

where $\boxed{\gamma'^2 = \gamma^2 - (\frac{\Delta k}{2})^2}$

derivation of (12.8)

$$\frac{d}{dz} \left| \frac{da_1}{dz} - \frac{i\Delta k}{2} a_1 = -i\gamma a_2^* \right.$$
$$\frac{d}{dz} \left| \frac{da_2}{dz} - \frac{i\Delta k}{2} a_2 = -i\gamma a_1^* \right.$$

$$\frac{d^2a_1}{dz^2} - \frac{i\Delta k}{2}\frac{da_1}{dz} = -i\gamma(-\frac{i\Delta k}{2}a_2^* + i\gamma a_1)$$

$$\frac{d^2a_1}{dz^2} - \frac{i\Delta k}{2}(\frac{i\Delta k}{2}a_1 - \cancel{i\gamma a_2^*}) = -i\gamma(-\cancel{\frac{i\Delta k}{2}}a_2^* + i\gamma a_1)$$

→ $\frac{d^2a_1}{dz^2} - (\gamma^2 - (\frac{\Delta k}{2})^2)a_1 = 0$

same for a_2

The solution is:

$$a_1 = \alpha_1 cosh(\gamma' z) + \beta_1 sinh(\gamma' z)$$
$$a_2 = \alpha_2 cosh(\gamma' z) + \beta_2 sinh(\gamma' z)$$

Optical parametric process with some phase mismatch

$$A_1 = \{\alpha_1 cosh(\gamma' z) + \beta_1 sinh(\gamma' z)\}e^{-i\Delta kz/2}$$

and

$$A_2 = \{\alpha_2 cosh(\gamma' z) + \beta_2 sinh(\gamma' z)\}\, e^{-i\Delta kz/2}$$

For initial conditions: $A_1 = A_{10}$; $A_2 = 0$

$$A_1 = \{A_{10} cosh(\gamma' z) + \beta_1 sinh(\gamma' z)\}e^{-i\Delta kz/2}$$

$$A_2 = \{\alpha_2 cosh(\gamma' z) + \beta_2 sinh(\gamma' z)\}\, e^{-i\Delta kz/2}$$

=0

Let us find β_1 and β_2

$$\frac{dA_1}{dz} = -i\gamma A_2^* e^{-i\Delta kz}$$

$$\frac{dA_2}{dz} = -i\gamma A_1^* e^{-i\Delta kz} \qquad (12.7)$$

left side

$$\frac{dA_1}{dz} = (-\frac{i\Delta kz}{2})\{A_{10}\underline{cosh(\gamma' z)} + \underline{\underline{\beta_1 sinh}}(\gamma' z)\}e^{-\frac{i\Delta kz}{2}} + \gamma'\ \{A_{10}\underline{\underline{sinh}}(\gamma' z) + \underline{\beta_1 cosh(\gamma' z)}\}e^{-\frac{i\Delta kz}{2}}$$

right side

$$-i\gamma A_2^* e^{-i\Delta kz} = (-i\gamma)\beta_2^*\ sinh(\gamma' z)\ e^{+i\Delta kz/2}\ e^{-i\Delta kz} = (-i\gamma)\underline{\underline{\beta_2^*\ sinh}}(\gamma' z)\ e^{-i\Delta kz/2}$$

$$(-\frac{i\Delta kz}{2})\beta_1 + \gamma'\ A_{10} = (-i\gamma)\beta_2^*\ ;$$

$$A_{10}\{(\frac{\Delta k}{2})^2\ \frac{1}{\gamma'} + \gamma'\ \} = (-i\gamma)\beta_2^*;$$

$$A_{10}\frac{(\frac{\Delta k}{2})^2 + \overbrace{\gamma'^2}^{\gamma^2}}{\gamma'} = (-i\gamma)\beta_2^*;$$

$$\beta_1 = iA_{10}\frac{\Delta k}{2\gamma'}$$

$$\beta_2 = iA_{10}^*\frac{\Delta k}{2\gamma'}$$

Optical parametric process

The final solution is:

$$A_1 = A_{10}\,[cosh(\gamma' z) + i\frac{\Delta k}{2\gamma'} sinh(\gamma' z)]e^{-i\Delta kz/2}$$

$$A_2 = -iA_{10}^{*}\,\frac{\gamma}{\gamma'}\sinh(\gamma' z)\,e^{-i\Delta kz/2} \qquad (12.9)$$

In terms of intensities:

$$I_1 = I_{10}\,|cosh(\gamma' z) + i\frac{\Delta k}{2\gamma'} sinh(\gamma' z)|^2$$

$$= I_{10}\,[cosh^2(\gamma' z) + \left(\frac{\Delta k}{2\gamma'}\right)^2 sinh^2(\gamma' z)]$$

$$I_2 = \frac{\omega_2}{\omega_1} I_{10}\left(\frac{\gamma}{\gamma'}\right)^2 sinh^2(\gamma' z) \qquad (12.10)$$

$$\gamma'^2 = \gamma^2 - (\frac{\Delta k}{2})^2$$

Optical parametric process

$$\gamma' = \sqrt{\gamma^2 - (\frac{\Delta k}{2})^2}$$

What happens if $\gamma^2 - (\frac{\Delta k}{2})^2 < 0$?

$\cosh(ix) = \cos(x)$
$\sinh(ix) = \sin(x)$

$\cosh \rightarrow \cos$

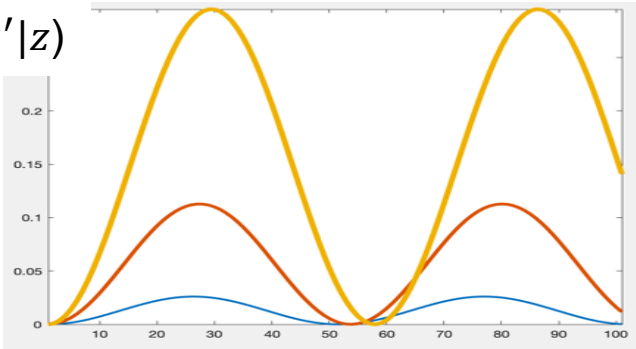
and

$\sinh \rightarrow \sin$

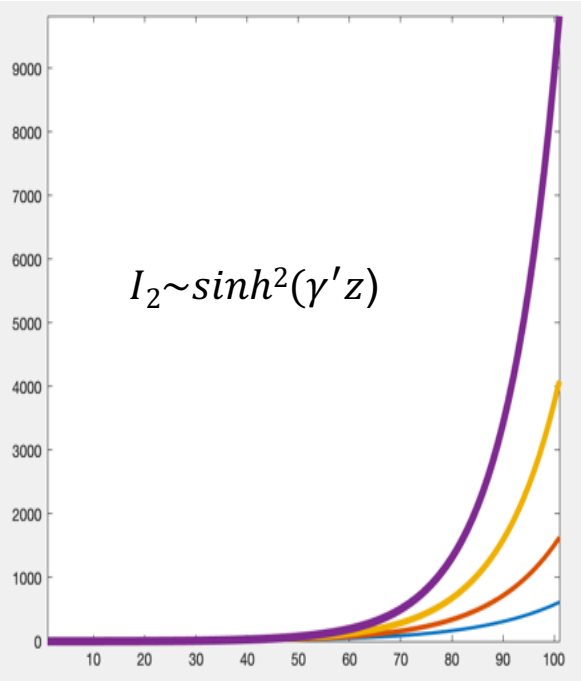
$$\gamma^2 - (\frac{\Delta k}{2})^2 < 0$$

periodic solution instead of steadily growing one

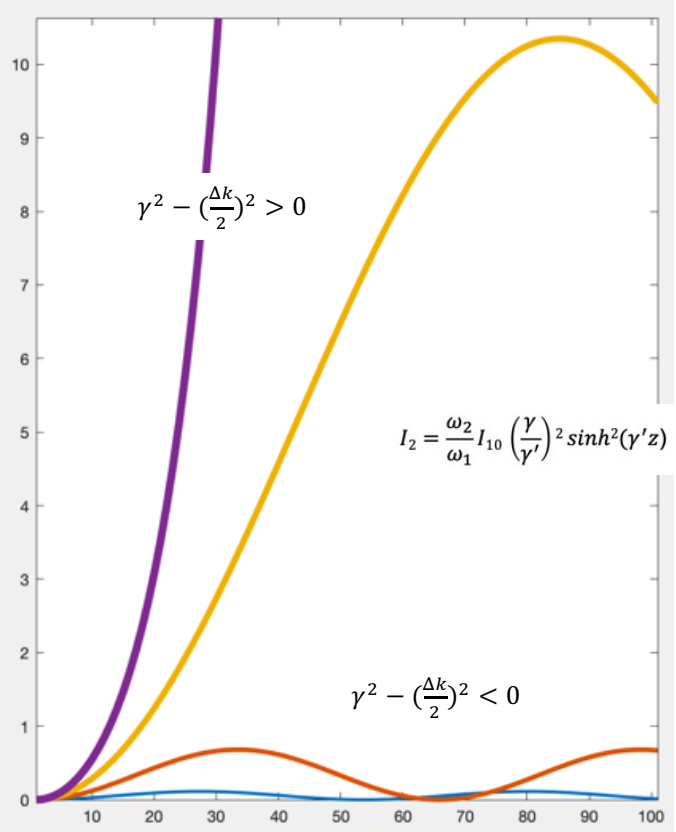
$I_2 \sim sin^2(|\gamma'|z)$



$$\gamma^2 - (\frac{\Delta k}{2})^2 > 0$$



Transition from periodic to exponentially growing one



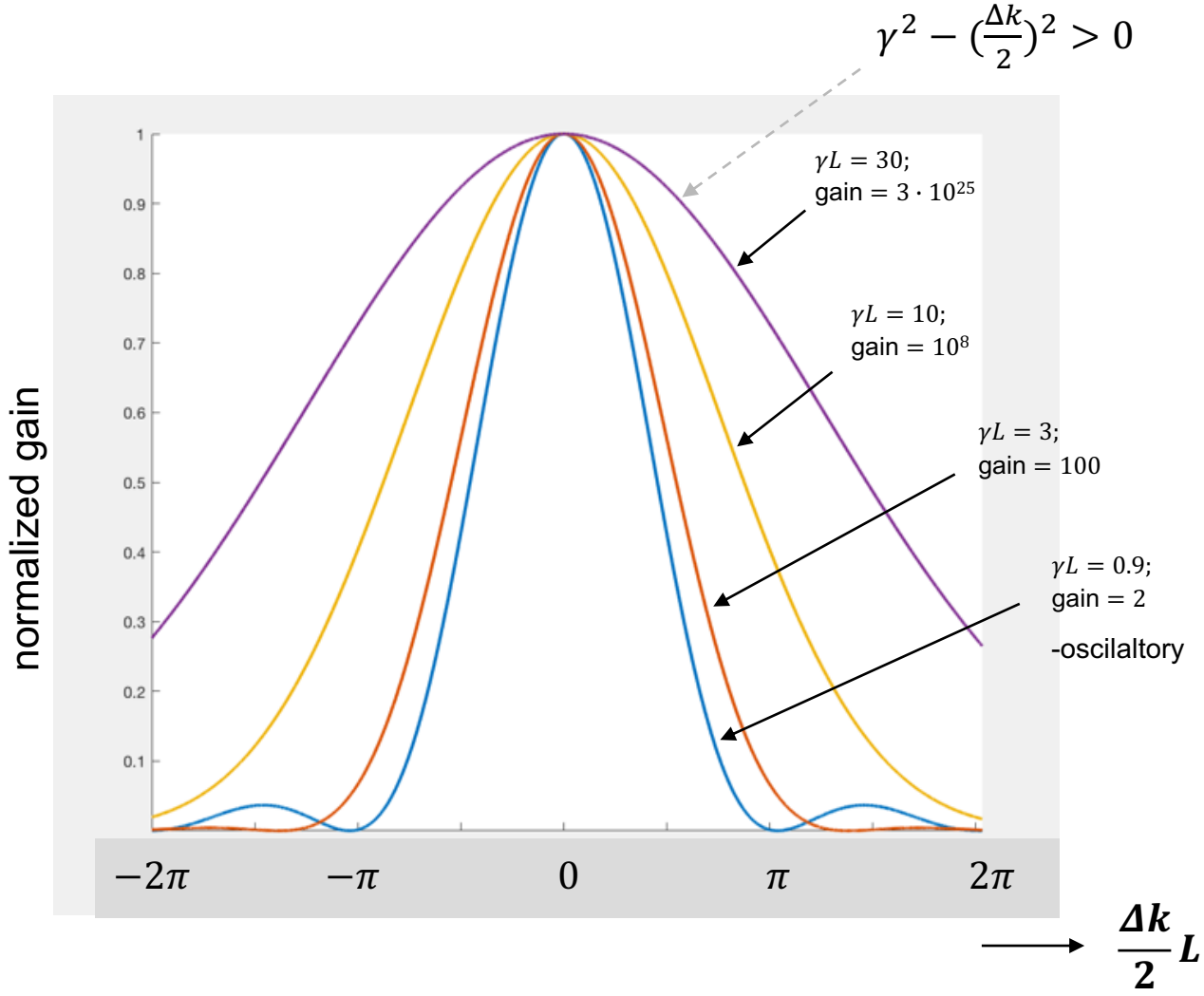
Optical parametric process

Less sensitive to phase mismatch at high gain

$$\gamma'^2 = \gamma^2 - (\frac{\Delta k}{2})^2$$

When the gain increment γ becomes large, the interaction becomes less sensitive to phase mismatch Δk

$$\text{gain} = \cosh^2(\gamma' \mathrm{L})$$



Optical parametric process for $\frac{\Delta k}{2} \gg \gamma$

Test (12.10) for $\frac{\Delta k}{2} \gg \gamma$ and $A_2 \ll A_{10} \ll A_3$ – Low efficiency DFG process to produce A_2

$$\gamma' = \sqrt{\gamma^2 - (\frac{\Delta k}{2})^2} \approx i\frac{\Delta k}{2} \qquad \gamma = gA_3 = \frac{d}{c}\sqrt{\frac{\omega_1\omega_2\omega_3}{n_1n_2n_3}}A_3 = \sqrt{\frac{2\omega_1\omega_2}{\varepsilon_0 c^3}(\frac{d^2}{n^3})\, I_{\omega_3}}$$

the 'usual' gain increment

$$\sinh(\gamma' z) \rightarrow \sin(\frac{\Delta k}{2}\mathrm{z})$$

From (12.10)

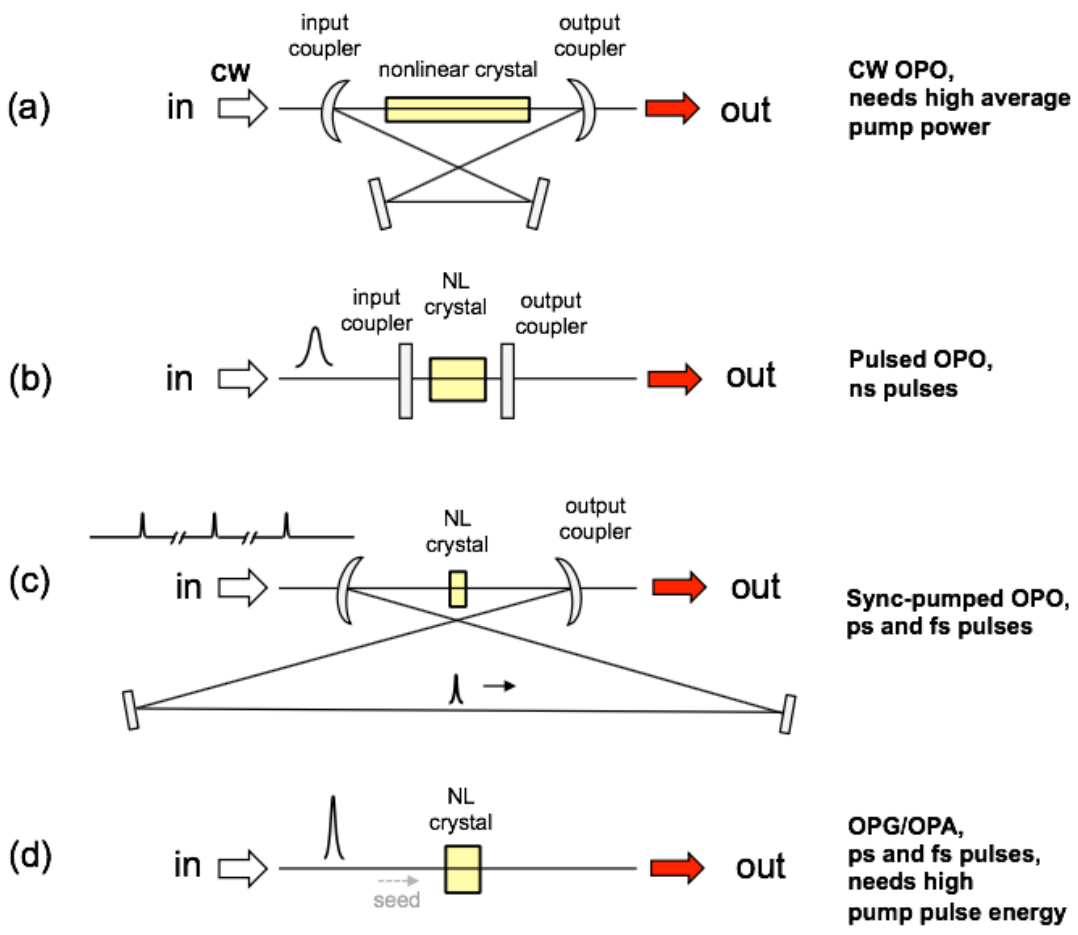
$$I_2 = \frac{\omega_2}{\omega_1} I_{10} |\frac{\gamma}{\gamma'}|^2 \sinh^2(\gamma' L) \rightarrow \frac{\omega_2}{\omega_1} I_{10} \left(\frac{\gamma}{\frac{\Delta k}{2}}\right)^2 \sin^2(\frac{\Delta k}{2} L) = \frac{\omega_2}{\omega_1} I_{10}\gamma^2 L^2 \frac{\sin^2(\frac{\Delta k}{2}L)}{(\frac{\Delta k}{2}L)^2} =$$

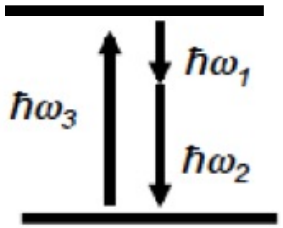
$$= \frac{\omega_2}{\omega_1} I_{10} \frac{2\omega_1\omega_2}{\varepsilon_0 c^3}\left(\frac{d^2}{n^3}\right) I_{\omega_3} L^2 \frac{\sin^2(\frac{\Delta k}{2}L)}{\left(\frac{\Delta k}{2}L\right)^2} = I_{10} I_{30} \frac{2{\omega_2}^2}{\varepsilon_0 c^3}\left(\frac{d^2}{n^3}\right) L^2 \,\mathrm{sinc}^2(\frac{\Delta k}{2}L)$$

- get the 'usual' formula for low efficiency DFG process with phase mismatch

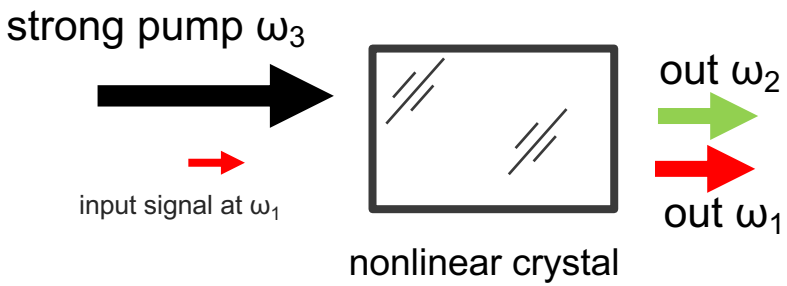
Optical parametric process

Four types of the optical parametric devices





Optical parametric amplifier (OPA)



$P_{out}(\omega_1) = G \times P_{in}(\omega_1)$

Parametric gain coefficient $G = \cosh^2(\gamma L)$

where $$\gamma = \sqrt{\frac{2\omega_1\omega_2}{\varepsilon_0 c^3}\left(\frac{d^2}{n^3}\right) I_{\omega_3}}$$ - on-axis gain increment

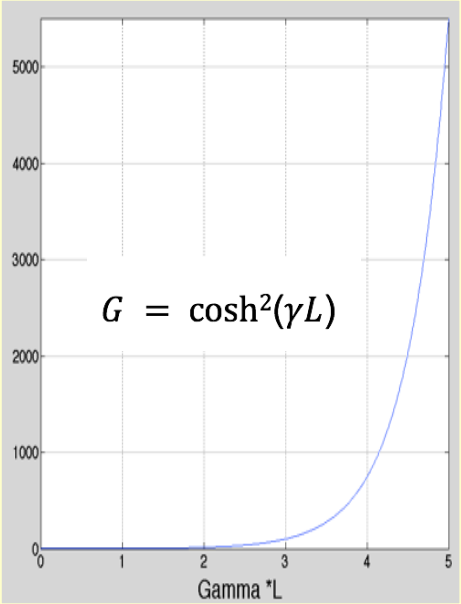
I_{ω_3} – pump intensity (power density) (W/cm^2)
$d=d_{eff}$ – nonlinear coefficient (pm/V)

fractional gain

For $(\gamma L) \ll 1$, $\cosh^2(\gamma L) \approx 1+(\gamma L)^2$

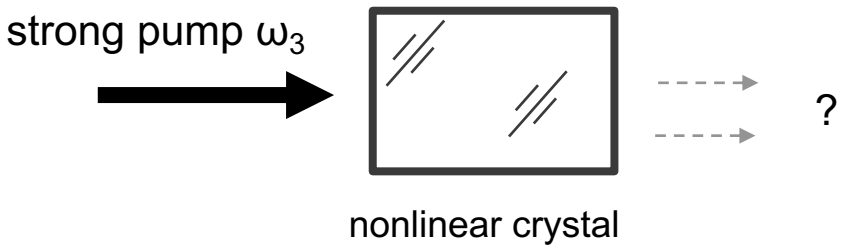
For $(\gamma L) \gg 1$, $\cosh^2(\gamma L) \approx \frac{1}{4}\exp(2\gamma L)$

cosh^2 is a fast growing function



Pump	Pump intensity	Parametric gain
CW laser, few W	~ 5 kW/cm^2	few %
Nanosecond pulses ~ 1 mJ/pulse	~ 1 MW/cm^2	~ 10
Intense *ps* or *fs* pulses ~ 1 mJ/pulse	~ 1 GW/cm^2	~ 10^{10}

Optical parametric generator (OPG)



Classical picture:
From (12.3) and (12.4) and it follows that if there is no input : $A_{10} = A_{20} = 0$, **the output is zero**

Quantum picture: The input is the vacuum state quantum noise : **one photon per mode**.

Quantum theory tells (see Yariv "Quant. Electron." book) that in fact that the amount of the output photons (per mode), n_1 and n_2 at ω_1 and ω_1, is given by

$$n_1 = n_{10}\cosh^2(\gamma L) + (1 + n_{20})sinh^2(\gamma L)$$

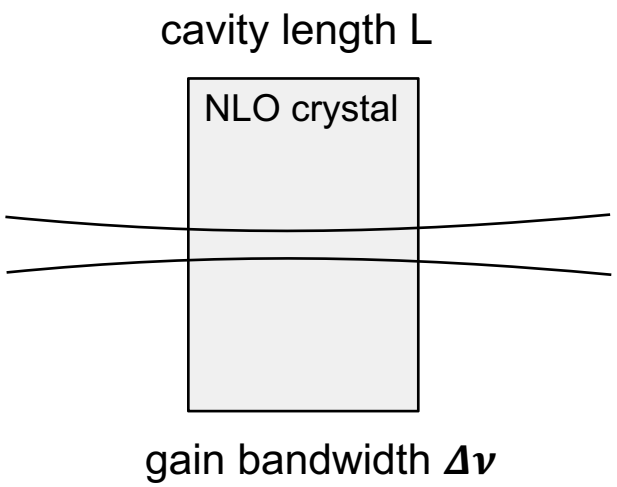
$$n_2 = n_{20}\cosh^2(\gamma L) + (1 + n_{10})sinh^2(\gamma L)$$

where n_{10} and n_{20} are the inputs at ω_1 and ω_1; and $\gamma = \sqrt{\frac{2\omega_1\omega_2}{\varepsilon_0 c^3}\left(\frac{d^2}{n^3}\right) I_{\omega_3}}$

Thus, for $A_{10} = A_{20} = 0$, the output is non-zero and is seeded by the **quantum noise**

Optical parametric generator and quantum noise

The quantum noise is associated with the vacuum state, which is equivalent to one photon per mode
How to make sense of one photon per mode?



Cavity length *L* → mode spacing $\Delta\nu_m = \frac{c}{2L}$

Total longitudianl modes (assume one transverse mode) $N = \Delta\nu/\Delta\nu_m = \Delta\nu\ 2L/c$

2L/c

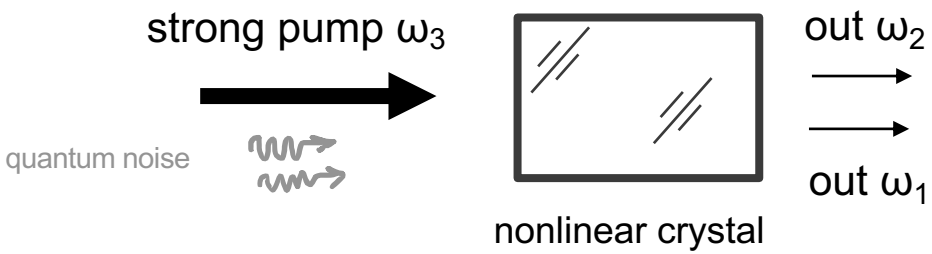
Total quantum noise 'power' entering the crystal = (total number of noise photons) x $\hbar\omega$ / (period of circulation) =

$$\boldsymbol{P_{Qnoise}} = N\hbar\omega\ /T = \Delta\nu\ 2L/c\ \hbar\omega/(2L/c) = \hbar\boldsymbol{\omega}\ \boldsymbol{x}\ \boldsymbol{\Delta\nu}$$

since *L* drops out, this is a universal formula

Example: $\Delta\nu$=1THz, $\lambda = 5\ \mu m\ (\hbar\omega = 4 \times 10^{-18}\ J)$ → $\boldsymbol{P_{Qnoise}} = \boldsymbol{40\ nW}$

Optical parametric generator (OPG)



High pumping intensity needed, typically > 1 GW/cm^2 – to achieve high (> 10^{10}) gain

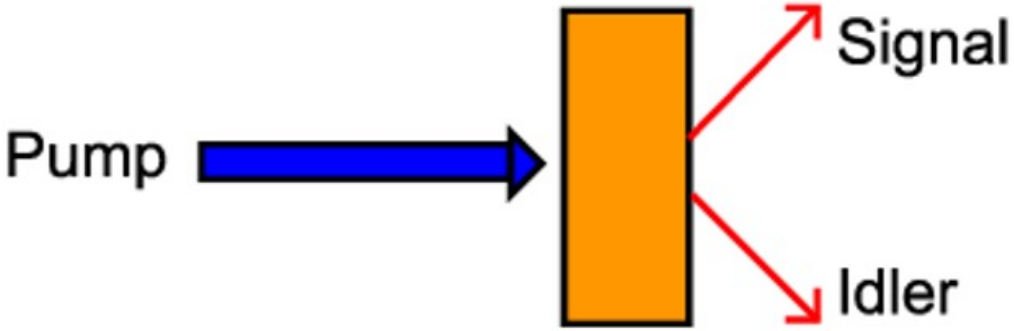
The principle of operation of a travelling-wave 'superfluorescent' optical parametric generator (OPG) is based on a single-pass high-gain (>10^{10}) amplification of quantum noise in a nonlinear crystal pumped by intense short laser pulses.

<u>The main characteristics of the travelling-wave OPGs:</u>

- Simplicity (no cavity needed). One pump laser.
- Ability to produce high peak power outputs (>1 MW) in the form of a single pulse.
- Broad tunability, restricted only by the phase-matching and crystal transparency.
- No resonant build-up time. This allows generating synchronized, independently tunable pulses for time-resolved spectroscopy.
- The output builds from quantim noise (~nW) to reach the peak power comparable to that of the pump (~ MW)

Spontaneous parametric down conversion as a source of entangled photons

Typically, low-power lasers (CW lasers) are used as a pump

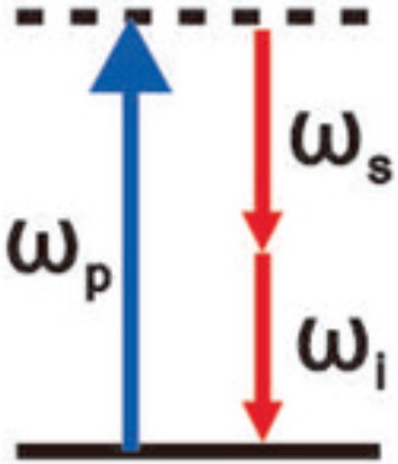


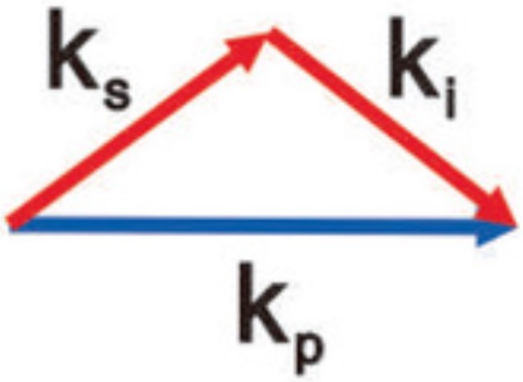
$$n_1 = n_{10}\cosh^2(\gamma L) + (1 + n_{20})sinh^2(\gamma L)$$

$$n_2 = n_{20}\cosh^2(\gamma L) + (1 + n_{10})sinh^2(\gamma L)$$

Spontaneous Parametric Down Conversion

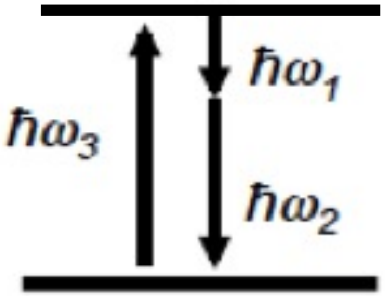
Energy conservation

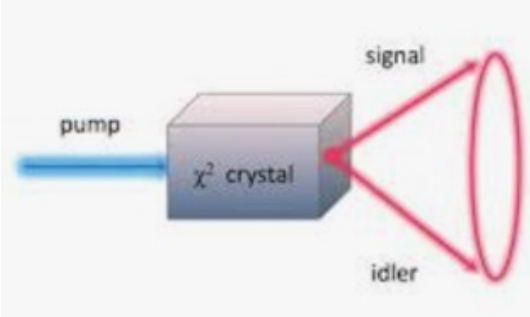


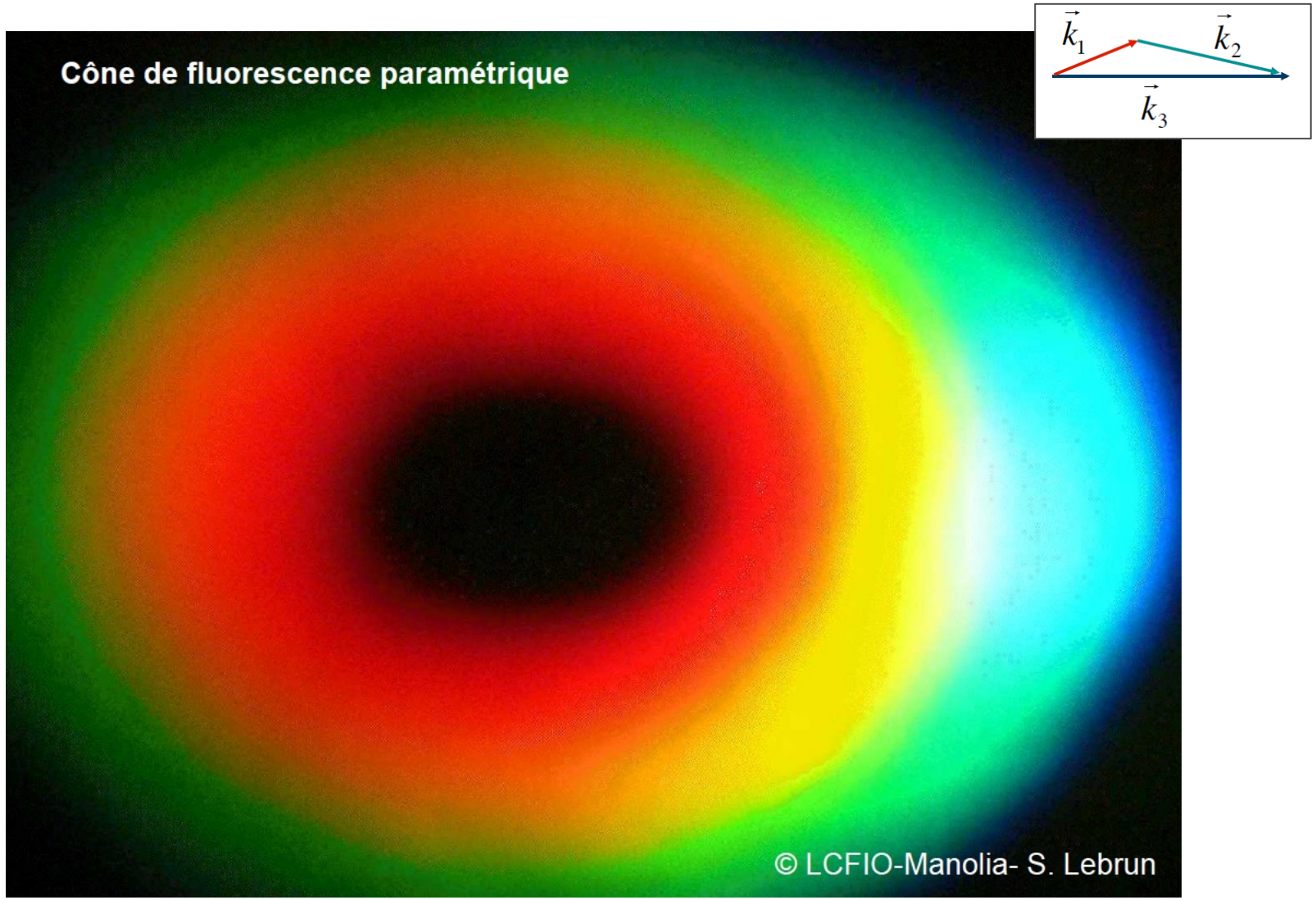


Momentum conservation

Optical parametric process



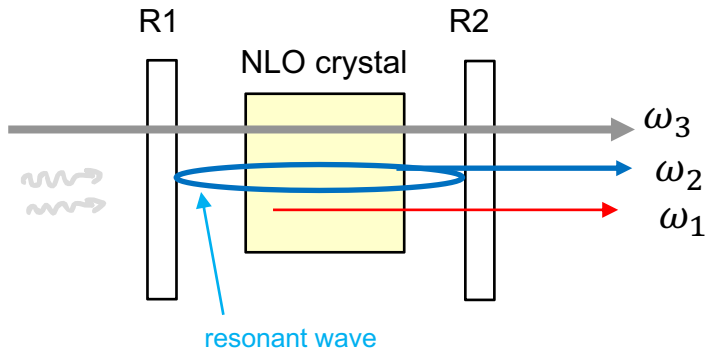




Optical parametric oscillator (OPO)

Singly -resonant OPO (SRO)

Let us find the oscillation threshold for a SRO. The OPO cavity is **resonant for ω_2**. Assume the fractional *intensity* roundtrip loss at resonating signal wave ω_2 to be $loss^I_{\omega 2} \ll 1$. For example it can be associated with non-unity reflection of the incoupling/outcoupling OPO mirrors ($loss^I_{\omega 2} = 1 - R_1 R_2$) or loss in the NLO crystal.



Then the threshold condition can be simply written as:

Gain

$$(1 - loss^I_{\omega 2}) \times \cosh^2(\gamma L) = 1;$$

fractional power gain per pass

For small OPO gain $(\gamma L)^2 \ll 1$

$$\cosh^2(\gamma L) \approx 1 + (\gamma L)^2$$

$$\gamma = g A_3$$

$$g = \frac{d}{c}\sqrt{\frac{\omega_1 \omega_2 \omega_3}{n_1 n_2 n_3}}$$

Assuming that the loss is small, $\ll 1$, and Δk=0, we can write

$$(1 - loss^I_{\omega 2})(1 + (\gamma L)^2) = 1$$

Simply put: the fractional gain per pass must equal to the fractional energy loss per pass.

or approximately

$$\boxed{(\gamma L)^2 = loss^I_{\omega 2}} \qquad (12.11)$$

replacing γ from (12.6)

$$\boxed{\frac{2\omega_1\omega_2}{\varepsilon_0 c^3}\left(\frac{d^2}{n^3}\right) I_{\omega_3} L^2 \approx loss^I_{\omega 2}}$$

or

$$\boxed{I^{thr}_{\omega_3} \approx \frac{\varepsilon_0 c^3}{2\omega_1\omega_2 L^2}\frac{loss^I_{\omega 2}}{(d^2/n^3)} \sim loss^I_{\omega 2}} \qquad (12.12)$$

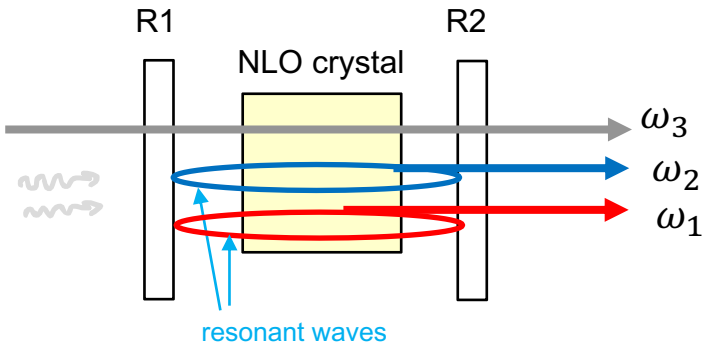
the threshold **intensity** is proportional to the loss at the resonating wave

Optical parametric oscillator (OPO)

Doubly -resonant OPO (DRO)

Now the OPO cavity is **resonant for *both* ω_1 and ω_2**.

Assume the fractional *intensity* roundtrip loss
at ω_1 is $loss^I_{\omega 1}$ so that the field loss is $loss^E_{\omega 1}=\frac{1}{2}loss^I_{\omega 1}$
at ω_2 is $loss^I_{\omega 2}$ so that the field loss is $loss^E_{\omega 2}=\frac{1}{2}loss^I_{\omega 2}$



Then the threshold condition can be found in this way:
As before in this Lecture, we write the coupled equations for the two resonating waves

$$\frac{dA_1}{dz} = -i\gamma A_2^* \qquad (5.7a)$$
$$\frac{dA_2}{dz} = -i\gamma A_1^*$$

in a DRO a fractional gain needed to achieve the threshold is very small, so we can assume that on a single pass $\Delta A \ll A$ and integrate these equations...

... to get field increase after passing the crystal

$$\Delta A_1 = -i\gamma L A_2^*$$
$$\Delta A_2 = -i\gamma L A_1^*$$

by adding the losses we get the field increase aftear one roundtrip

$$\Delta A_1 = -loss^E_{\omega 1} A_1 - i\gamma L A_2^*$$
$$\Delta A_2 = -i\gamma L A_1^* - loss^E_{\omega 2} A_2$$

The threshold condition is when $\Delta A_1 = \Delta A_2 = 0$

$$0 = -loss^E_{\omega 1} A_1 - i\gamma L A_2^*$$
$$0 = -i\gamma L A_1^* - loss^E_{\omega 2} A_2 \quad \Big|\ *$$

$$\Rightarrow \quad loss^E_{\omega 1} A_1 + i\gamma L A_2^* = 0$$
$$i\gamma L A_1 - loss^E_{\omega 2} A_2^* = 0$$

Optical parametric oscillator (OPO)

Doubly -resonant OPO (DRO)

For nontrivial solution the determinant $\begin{vmatrix} loss^E_{\omega 1} + i\gamma L & \\ i\gamma L & -loss^E_{\omega 2} \end{vmatrix}$ should $= 0$, so we get

DRO $$\boxed{(\gamma L)^2 = loss^E_{\omega 1} loss^E_{\omega 2} = \frac{1}{4} loss^I_{\omega 1} loss^I_{\omega 2}} \quad (12.13)$$

compare: SRO $$\boxed{(\gamma L)^2 = loss^I_{\omega 2}} \quad (12.11)$$

Comparing (12.11) and (12.13) one can see that the DRO threshold $I^{thr}_{\omega_3} \sim (\gamma L)^2$ – can be dramatically lower (by $\frac{4}{loss^I_{\omega 1}}$ times) than that for SRO

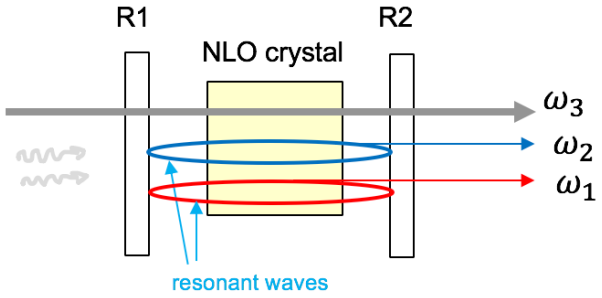
Example: for the fractional loss (in intensity) at ω_1 $loss^I_{\omega 1}$=4%, the DRO threshold is **100 times smaller** than for SRO

However in DRO both signal and idler waves have to belong to the cavity modes **simultaneously** (an **overconstrained** system). DRO is very sensitive (interferometrically) to the tiny cavity length fluctuations and hence is less preferrd in practical applications.

Optical parametric process

So why there is such a big difference in thresholds?

DRO: both waves are coherent and resonate



SRO: the idler wave (ω_1) does not resonate and every roundtrip has to start from scratch