Lecture 2

Math basics, vector algebra, Fourier transform, wave propagation equation, slowly varying envelope approximation (SVEA)

MATH BASICS

Math basics

(vector calculus)

In the differential equations,

- the nabla symbol, ∇, denotes the three-dimensional gradient operator, del,
- the $\nabla \cdot$ symbol (pronounced "del dot") denotes the divergence operator,
- the $\nabla \times$ symbol (pronounced "del cross") denotes the curl operator.

$$\nabla = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$\text{Gradient: grad } f = \overset{\text{vector}}{\nabla} \, \overset{\text{scalar}}{f} \qquad \text{=vector}$$

$$\text{Divergence: div } \vec{v} = \overset{\text{vector}}{\nabla} \cdot \overset{\text{vector}}{\vec{v}} \qquad \text{=scalar}$$

$$\text{Curl: curl } \vec{v} = \underset{\text{vector}}{\nabla} \times \underset{\text{vector}}{\vec{v}} \qquad \text{=vector}$$

Basically can treat 'nabla' as a vector

Math basics

Divergence (a scalar)

$$\operatorname{div}\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \nabla \cdot \vec{v}$$

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$$

vector

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

scalar

Math basics

Curl

$$\operatorname{curl}\vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\vec{e}_z = \nabla\times\vec{v}$$

$$\nabla\times\vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

Math basics

Cross product of 3 vectors

(curl of the curl identity)

$$\mathbf{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\boldsymbol{B})$$

$$\nabla\times(\nabla\times\boldsymbol{E}) = \underset{\text{vector}}{\nabla}(\underset{\text{scalar}}{\nabla\boldsymbol{E}}) - \underset{\text{scalar}}{\nabla^{2}}\underset{\text{vector}}{\boldsymbol{E}} \tag{2.1}$$

Math basics

Fourier Transform

We will use this definition (see e.g. Weiner “Ultrafast Optics” or Keller “Ultrafast lasers”)

$$\tilde{f}(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$f(t) = \mathcal{F}^{-1}\{\tilde{f}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega \qquad (2.2)$$

Whatever definition you use, it should always be:

$$\mathcal{F}^{-1}\{\mathcal{F}\{f(t)\}\} = f(t)$$

There is a plethora of rival conventions on the definition of the Fourier transform

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$$\tilde{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, d\omega$$

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega$$

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t}\, d\omega$$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, d\omega$$

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega$$

Math basics

Fourier Transform

Derivatives

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega$$

$$\frac{d}{dt}f(t) = \frac{d}{dt}\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t} d\omega\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\boldsymbol{i\omega}\tilde{f}(\omega)}_{\text{new Fourier transform}}\, e^{i\omega t} d\omega$$

$$\frac{d^2}{dt^2}f(t) = \frac{d^2}{dt^2}\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t} d\omega\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{-\boldsymbol{\omega^2}\tilde{f}(\omega)}_{\text{new Fourier transform}}\, e^{i\omega t} d\omega$$

$$\boxed{\begin{aligned} &\mathcal{F}\left\{\frac{d}{dt}f(t)\right\} = \boldsymbol{i\omega}\tilde{f}(\omega) \\ &\mathcal{F}\left\{\frac{d^2}{dt^2}f(t)\right\} = -\boldsymbol{\omega^2}\tilde{f}(\omega) \\ &\mathcal{F}^{-1}\{\boldsymbol{i\omega}\tilde{f}(\omega)\} = \frac{d}{dt}f(t) \\ &\mathcal{F}^{-1}\{-\boldsymbol{\omega^2}\tilde{f}(\omega)\} = \frac{d^2}{dt^2}f(t) \end{aligned}} \quad (2.3)$$

MAXWELL’S EQUATIONS

Maxwell's equations

(in the medium)

$$\nabla \cdot \boldsymbol{D} = \rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{d\boldsymbol{B}}{dt}$$

$$\nabla \times \boldsymbol{H} = \frac{d\boldsymbol{D}}{dt} + \boldsymbol{J}$$

- Coulomb's law
- Faraday's law of induction
- Ampère's circuital law
- Helmholz equation in optics
- etc.....

E - electric field

D - electric displacement field

H - magnetizing field

B - magnetic field

in optics:

no free charges: ρ=0

no free currents: $\boldsymbol{J}$=0

the material is nonmagnetic: $\boldsymbol{B} = \mu_0 \boldsymbol{H}$

$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

$\boldsymbol{P}$ - Polarization of the material = dipole moment per unit volume.

Constants

Permeability in vacuum (air) $\mu_0 = 4\pi \times 10^{-7} \quad H/m$ = 1.257 x 10^{-6} H/m

Permittivity in vacuum (air) $\varepsilon_0 = \dfrac{1}{\mu_0 c^2}$ = 8.85 x 10−12 F/m

$$\text{the speed of light, } c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \qquad = 3 \times 10^8 \text{ m/c}$$

$$\mu_0 = \frac{1}{c^2 \varepsilon_0}$$

good to remember !

impedance of the free space $\eta = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 377\ \Omega$

WAVE PROPAGATION

Wave propagation

start from Maxwell's equations

$$\nabla\times \quad \boxed{\begin{aligned} &\nabla \cdot \boldsymbol{D} = \boldsymbol{0} \\ &\nabla \cdot \boldsymbol{B} = 0 \\ &\nabla\times\boldsymbol{E} = -\frac{d\boldsymbol{B}}{dt} \\ &\nabla\times\boldsymbol{H} = \frac{d\boldsymbol{D}}{dt} \end{aligned}}$$

$$B = \mu_0 H$$

$$\nabla\times\nabla\times\boldsymbol{E} = -\nabla\times\left(\frac{d\boldsymbol{B}}{dt}\right) = -\frac{d}{dt}(\nabla\times\boldsymbol{B}) = -\mu_0\frac{d}{dt}(\nabla\times\boldsymbol{H}) = -\mu_0\frac{d}{dt}\left(\frac{d\boldsymbol{D}}{dt}\right) = -\mu_0\frac{d^2\boldsymbol{D}}{dt^2}$$

on the other hand:

$$\nabla\times\nabla\times\boldsymbol{E} = \nabla(\nabla\boldsymbol{E}) - \nabla^2\boldsymbol{E} \quad = -\nabla^2\boldsymbol{E}$$

(curl of the curl identity)

$$\mathbf{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{AC}) - \boldsymbol{C}(\boldsymbol{AB})$$

= 0 since $\nabla\boldsymbol{E} = \nabla\boldsymbol{D}/\varepsilon = 0$

$$\boxed{\nabla^2\boldsymbol{E} - \mu_0\frac{d^2\boldsymbol{D}}{dt^2} = 0}$$

Wave propagation

Using

$\boldsymbol{D} = \varepsilon \boldsymbol{E}$

finally get:

$$\nabla^2 \boldsymbol{E} - \mu_0 \frac{d^2 \boldsymbol{D}}{dt^2} = 0 \quad \rightarrow \quad \nabla^2 \boldsymbol{E} - \varepsilon\mu_0 \frac{d^2 \boldsymbol{E}}{dt^2} = 0 \quad \rightarrow \quad \nabla^2 \boldsymbol{E} - \frac{n^2}{c^2}\frac{d^2 \boldsymbol{E}}{dt^2} = 0 \qquad (2.4)$$

$\mathrm{c} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ - speed of light in vacuum

Useful relations:

$$D = \varepsilon E = n^2 \varepsilon_0 E = \varepsilon_0 E + P$$
$$P = \varepsilon_0 \chi E$$
$$1 + \chi = n^2 = \varepsilon / \varepsilon_0$$

n - ref. index

P - polarization of a dielectric; χ - linear susceptibility,

relative permittivity

In vacuum: $$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{d^2 \boldsymbol{E}}{dt^2} = 0 \qquad (2.5)$$

Wave equation
- same for optics, acoustics etc..

Compact form for wave equation (u -scalar) $\ddot{u} = c^2 \nabla^2 u$

The wave equation and plane waves

$$\nabla^2 \boldsymbol{E} = \frac{d^2\boldsymbol{E}}{dx^2} + \frac{d^2\boldsymbol{E}}{dy^2} + \frac{d^2\boldsymbol{E}}{dz^2}$$

is a vector equation

for plane transverse wave propagating <u>along z</u>, with ***E*** -> E_{xy} (scalar, xy plane)

$$\nabla^2 \boldsymbol{E} - \frac{n^2}{c^2}\frac{d^2\boldsymbol{E}}{dt^2} = 0 \longrightarrow \frac{d^2 E}{dz^2} - \frac{n^2}{c^2}\frac{d^2 E}{dt^2} = 0 \qquad (2.6)$$

solution for monochromatic laser radiation :

$$\underbrace{E = Ae^{i\omega t - ikz}}_{\Longrightarrow} \quad \text{or} \quad \underbrace{E = Ae^{i\omega t + ikz}}_{\Longleftarrow}$$

A - const (can be complex)

where $k=\omega n/c$ and phase velocity is $\omega/k = c/n$

in fact, the 'physical' field for the forward moving wave is

$$\frac{1}{2}\left(Ae^{i\omega t - ikz} + c.c.\right)$$

complex conjugate

Energy density, intensity vs. field amplitude

$$E(t) = E cos\omega t, \qquad (E - peak\ \ amplitude)$$

In free space

In a medium

The average optical **energy density** and field amplitude E are related as:

[J/m^3] $$U = \frac{1}{2}\varepsilon_0 |E|^2$$

$$U = \frac{1}{2}\varepsilon |E|^2 = \frac{1}{2}\varepsilon_0 n^2 |E|^2 \qquad (2.7)$$

The average optical **intensity** and field amplitude E are related as:

[W/m^2] $$I = \frac{1}{2} c\varepsilon_0 |E|^2 = |E|^2/2\eta_0$$

$$I = \frac{1}{2}(c/n)\varepsilon_0 n^2 |E|^2 = \frac{1}{2} cn\varepsilon_0 |E|^2 = |E|^2/2\eta \qquad (2.8)$$

η_0 is the impedance of the free space

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega$$

η is the characteristic impedance of the of the medium

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon}} = \eta_0/n$$

see e.g. Salech, Teich book

Slowly varying envelope approximation

Slowly varying envelope approximation (SVEA)

dielectric without any external sources $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \varepsilon \boldsymbol{E}$

add a time-varying polarization that act as a source of new components of the electromagnetic field (perturbation polarization)

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} + \boldsymbol{P}_{ext}$$

Plug this new ***D*** into eq. (2.4) :

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu_0 \frac{d^2 \boldsymbol{D}}{dt^2} = -\frac{n^2}{c^2}\frac{d^2 \boldsymbol{E}}{dt^2} - \mu_0 \frac{d^2 \boldsymbol{P}_{ext}}{dt^2}$$

$$-\nabla^2 \boldsymbol{E} = -\frac{d^2 E}{dz^2}$$

for plane -wave approx.

Scalar wave equation:

$$\boxed{\frac{\partial^2 E}{\partial z^2} - \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{ext}}{\partial t^2}} \qquad (2.9)$$

Slowly varying envelope approximation (SVEA)

for harmonic waves, look for a solution : $E = E(z)e^{i\omega t-ikz}$, $E(z)$ **- varies slowly**

left side of (2.9):

small

$$\frac{\partial^2 E}{\partial z^2} = e^{i\omega t-ikz}\{(-ik)^2E(z) - 2ik\frac{\partial E(z)}{\partial z} + \frac{\partial^2 E(z)}{\partial z^2}\} = e^{i\omega t-ikz}(-k^2E(z) - 2ik\frac{\partial E(z)}{\partial z} + \frac{\partial^2 E(z)}{\partial z^2})$$

+

$$-\frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} = (-\frac{n^2}{c^2})e^{i\omega t-ikz}(i\omega)^2\, E(z) = e^{i\omega t-ikz}\frac{n^2}{c^2}\omega^2\, E(z) = e^{i\omega t-ikz}k^2\, E(z)$$

SVEA equation:

$$\boxed{2ik\,\frac{\partial E(z)}{\partial z}\, e^{i\omega t-ikz} = -\mu_0\frac{\partial^2 P}{\partial t^2}} \qquad (2.10)$$

perturbation polarization

slowly varying E-filed

Slowly varying envelope approximation (SVEA)

for harmonic perturbation polarization moving at the same phase velocity and having the same frequency ω:

$$P(z,t) = Pe^{i\omega t - ikz}$$

$$2ik\,\frac{\partial E}{\partial z}\,e^{i\omega t - ikz} = -\mu_0 \frac{\partial^2 P(z,t)}{\partial t^2}$$

$\{\frac{\partial^2}{\partial t^2} \rightarrow (\mathrm{i}\omega)^2$ -> $-\omega^2\}$

$$2ik\,\frac{\partial E}{\partial z}\,e^{i\omega t - ikz} = -\mu_0(-\omega^2)Pe^{i\omega t - ikz}$$

$$\frac{\partial E}{\partial z} = \frac{1}{2ik}\,\mu_0\omega^2 P = \frac{1}{2i\omega n/c}\,\mu_0\omega^2 P = -\frac{i\omega c}{2n}\,\mu_0 P$$

$$\boxed{\frac{\partial E(z)}{\partial z} = -\frac{i\omega c}{2n}\,\mu_0 P = -\frac{i\omega}{2\varepsilon_0 cn}\,P} \qquad (2.11)$$

perturbation polarization

the speed of light, $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$

$\mu_0 = \frac{1}{\varepsilon_0 c^2}$

Comment on the choice of the complex notation

$E \sim\ e^{i\omega t - ikz}$ vs. $E \sim\ e^{ikz - i\omega t}$

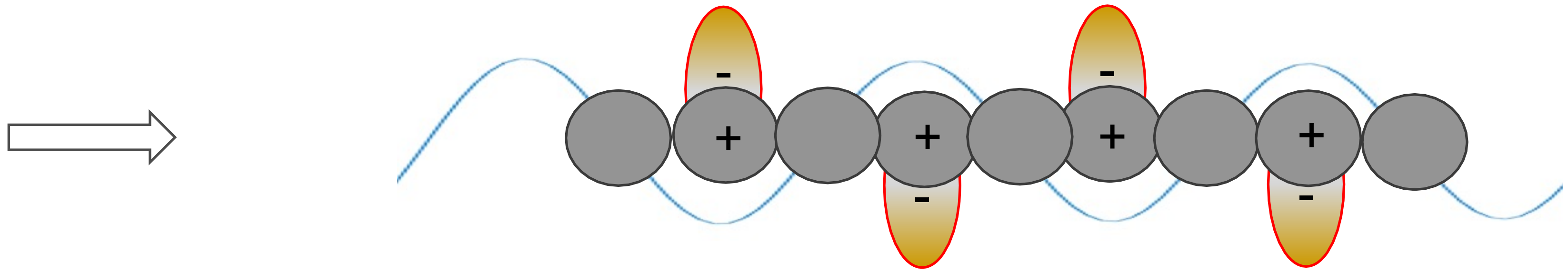
$$\boxed{\frac{\partial E(z)}{\partial z} = -\frac{i\omega c}{2n}\ \mu_0 P}$$

$$\boxed{\frac{\partial E(z)}{\partial z} = +\frac{i\omega c}{2n}\ \mu_0 P}$$

The SVEA equations (2.11) look different (but the final result should be the same)

Wave propagation

Example: The Lorentz oscillator model



A monochromatic plane electromagnetic wave is incident on a transparent dielectric. Assume that the electrons in the medium can be represented by Lorentz oscillators—classical charged harmonic oscillators, governed by the equation of motion:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = qE(t)/m \qquad (2.11)$$

Here $q = -e$ and m are the charge and mass of the electron, ω_0 is the fundamental frequency of the oscillator, and γ is a damping constant associated with the loss of energy by radiation or collisions.

For a monochromatic wave $E = Ee^{i\omega t}$ we get $(-\omega^2 + i\omega\gamma + \omega_0^2)x = eE/m$

so that $x = \dfrac{eE/m}{(\omega_0^2 - \omega^2 + i\omega\gamma)}$

The induced dipole moment $= ex$
Polarization P of the material = dipole moment per unit volume.

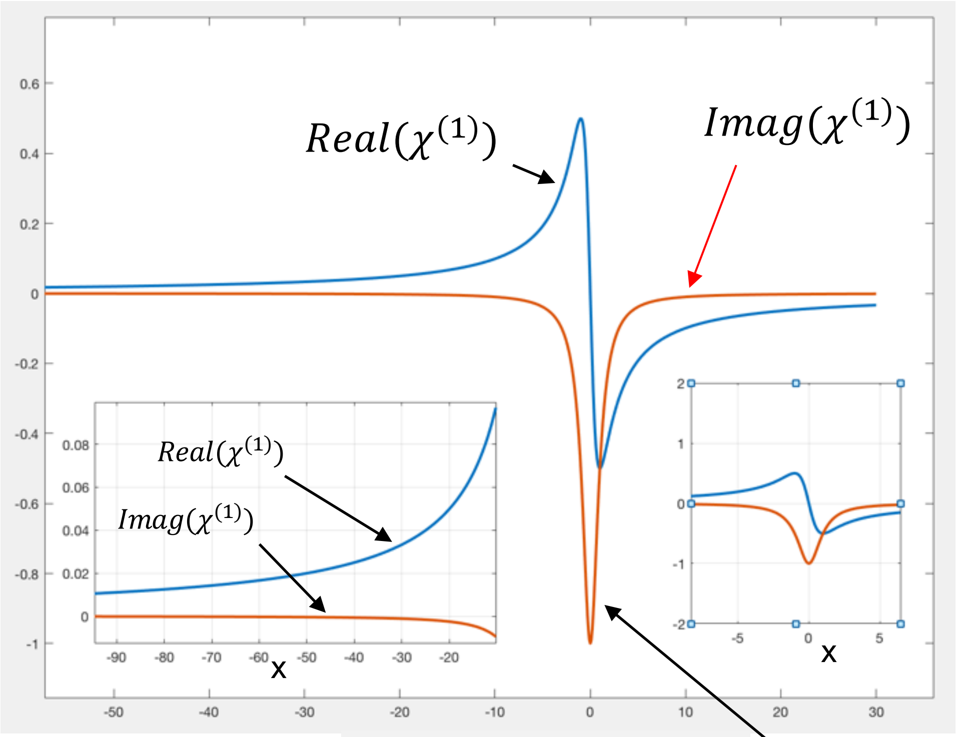
$$P = eNx = \frac{e^2N/m}{(\omega_0^2 - \omega^2 + i\omega\gamma)}E \qquad (2.12)$$

(N - number density of dipoles)

Linear susceptibility χ$^{(1)}$, classical harmonic oscillator

From the definition of linear susceptibility $\chi^{(1)}$: $\quad P = \varepsilon_0 \chi^{(1)} E$

we get $\quad \chi^{(1)} = \dfrac{Nq^2/m}{\epsilon_0(\omega_0^2 - \omega^2 + i\omega\gamma)}$



$x = (\omega - \omega_0)/(\gamma/2)$

=distance from resonance

$Imag(\chi^{(1)})$ is negative

$$Real(\chi^{(1)}) = \frac{Nq^2}{\epsilon_0 m}\frac{(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}$$

$$Imag(\chi^{(1)}) = -\frac{Nq^2}{\epsilon_0 m}\frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \rightarrow -\frac{Nq^2}{\epsilon_0 \omega_0 m}\frac{1}{\gamma}$$

at resonance $\omega = \omega_0$

Away from resonance:

$Real(\chi^{(1)}) \sim \dfrac{1}{(\omega_0 - \omega)}$ decays slow

$Imag(\chi^{(1)}) \sim \dfrac{1}{(\omega_0 - \omega)^2}$ decays faster

Wave propagation

The Lorentz oscillator model

Now plug P into eq. (2.3)

$$\frac{\partial E(z)}{\partial z} = -\frac{i\omega c}{2n}\ \mu_0 P = -\frac{i\omega c}{2n}\ \mu_0 \frac{e^2 N/m}{(\omega_0^2 - \omega^2 + i\omega\gamma)} E(z)$$

1) Assume $\omega=\omega_0$ (**resonance**)

$$\frac{\partial E(z)}{\partial z} = -\frac{i\omega c}{2n}\ \mu_0 \frac{e^2 N}{im\omega\gamma} E(z) = -\frac{c}{2n}\ \mu_0 \frac{e^2 N}{m\gamma} E(z)$$

property of the medium

→ $$\frac{\partial E}{\partial z} = -\alpha E$$ where $$\alpha = \frac{\mu_0 c e^2 N}{2nm\gamma} = \frac{1}{2c\varepsilon_0 n} \frac{e^2 N}{m\gamma}$$

the speed of light, $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

Solution: $$\boxed{E(z) = E_0 e^{-\alpha z}}$$

$$\boxed{I \sim E E^*; \quad I = I_0 e^{-2\alpha z}}$$

Wave propagation

The Lorentz oscillator model

2) Assume $\omega_0^2 - \omega^2 >> \omega\gamma$ (off resonance on the **lower frequency side**)

the speed of light, $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$$\frac{\partial E(z)}{\partial z} = -\frac{i\omega c}{2n}\mu_0 P = -\frac{i\omega c}{2n}\mu_0 \frac{e^2 N/m}{(\omega_0^2 - \omega^2)} E(z)$$

$$\mu_0 = \frac{1}{\varepsilon_0 c^2}$$

→ $$\frac{\partial E}{\partial z} = -i\beta E \quad \text{where} \quad \beta = \frac{\omega \mu_0 c}{2n}\frac{e^2 N/m}{(\omega_0^2 - \omega^2)} = \frac{\omega}{2c\varepsilon_0 n}\frac{e^2 N/m}{(\omega_0^2 - \omega^2)}$$

Solution: $$\boxed{E(z) = E_0(z) e^{-i\beta z}}$$

Only phase delay of the light beam;

Beam intensity ~ $|E_0|^2 = EE^* = const$